Solutions for Preparation Exam Advanced Econometrics (4.1)

Master Econometrics and Operations Research
School of Business and Economics

Exam:	Advanced Econometrics (4.1)
Code:	E_EORM_AECTR
Coordinator:	F. Blasques
Date:	–
Time:	–
Duration:	2 hours and 45 minutes
Calculator:	Not allowed
Graphical calculator:	Not allowed
Number of questions:	4
Type of questions:	Open
Answer in:	English
Credit score:	100 credits counts for a 10
Grades:	Made public within 10 working days
Inspection:	By appointment (send e-mail to f.blasques@vu.nl)
Number of pages:	5, including front page

- Read the entire exam carefully before you start answering the questions.
- Be clear and concise in your statements, but justify every step in your derivations.
- If you think that further information is needed to answer a question or that the question is ill-posed, then explain your reasoning.
- The questions should be handed back at the end of the exam. Do not take it home.

Good luck!

Question 1 [28 points] Multiple Choice

For each of the following multiple choice questions, please indicate which statement is correct by selecting option (a), (b), (c), or (d), or alternatively, if none of the options are correct, please state 'No answer is correct'.

All answers should be written on the answer sheet clearly and in order.

Note: You get 4 points for each correct answer and -1 point for every incorrect answer.

Note: If you believe that more than one statement is correct, then please indicate which answers are correct and why.

Note: Please write your answers in the answer sheet. Clearly indicate your choice. No justifications are needed.

1. Consider the following Asymmetric GARCH model

$$x_t = \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1),$$
$$\text{where} \quad \sigma_t^2 = \omega + \alpha x_{t-1}^2 + \delta x_{t-1} + \beta \sigma_{t-1}^2 \quad \text{for every} \quad t \in \mathbb{Z}.$$

   (a) A positive parameter $\delta > 0$ can be used to account for the *'Leverage Effect'*. If $\delta > 0$, then past negative returns $x_{t-1} < 0$ tend to have a smaller effect on the conditional volatility σ_t^2 than positive returns $x_{t-1} > 0$ of equal magnitude.

   (b) A negative parameter $\delta < 0$ can be used to account for the *'Leverage Effect'*. If $\delta < 0$, then past negative returns $x_{t-1} < 0$ tend to have a smaller effect on the conditional volatility σ_t^2 than positive returns $x_{t-1} > 0$ of equal magnitude.

   (c) A positive parameter $\delta > 0$ can be used to account for the *'Leverage Effect'*. If $\delta > 0$, then past negative returns $x_{t-1} < 0$ tend to have a larger effect on the conditional volatility σ_t^2 than positive returns $x_{t-1} > 0$ of equal magnitude.

   (d) A negative parameter $\delta < 0$ can be used to account for the *'Leverage Effect'*. If $\delta < 0$, then past negative returns $x_{t-1} < 0$ tend to have a larger effect on the conditional volatility σ_t^2 than positive returns $x_{t-1} > 0$ of equal magnitude.

2. Let $\{x_t\}_{t\in\mathbb{Z}}$ be a strictly stationary and ergodic sequence with two bounded moments $\mathbb{E}|x_t|^2 < \infty$. Let $y_t = 5x_t^2$ the sample average $\frac{1}{T}\sum_{t=1}^{T} y_t$.

   (a) The sample average converges to the expectation $\mathbb{E}(x_t)$ by application of a law of large numbers.

   (b) The sample average may or may not converge to the expectation $\mathbb{E}(x_t)$.

   (c) The sample average converges to the expectation $\mathbb{E}(5x_t^2)$ by application of a law of large numbers.

   (d) The sample average does not converge to the expectation $\mathbb{E}(5x_t^2)$.

Question 1 [28 points] - Multiple Choice (continued)

3. Consider the following regression model,

$$y_t = \alpha + \beta x_t + \varepsilon_t.$$

Suppose that you wish to perform structural econometric analysis, but the regressor x_t is endogenous, i.e. $\mathbb{E}(\varepsilon_t | x_t) \neq 0$. For this reason, you consider using another variable z_t as an instrument in the following regression

$$x_t = \delta + \gamma z_t + v_t.$$

Luckily, you find that the variable z_t is highly correlated with x_t, so it useful in predicting the endogenous regressor x_t.

(a) If z_t is independent of ε_t, then z_t is a *valid instrument.*

(b) If z_t is independent of both ε_t and v_t, then z_t is a *valid instrument.*

(c) If z_t is independent of v_t, then z_t is a *valid instrument.*

(d) If z_t is independent of both ε_t and y_t, then z_t is. a *valid instrument.*

4. Let $\{x_t\}_{t\in\mathbb{Z}}$ be a strictly stationary and ergodic sequence of stock returns with four bounded moments $\mathbb{E}|x_t|^4 < \infty$. Consider the following GARCH filtering equation for a the conditional volatility with $\boldsymbol{\theta} := (\omega, \alpha, \beta) = (0.1, 0.15, 0.95)$,

$$\sigma_t^2 = 0.1 + 0.15x_{t-1}^2 + 0.95\sigma_{t-1}^2 \quad \text{for every} \quad t \in \mathbb{Z}.$$

(a) The filtered volatility $\{\hat{\sigma}_t^2(\boldsymbol{\theta}, \hat{\sigma}_1^2)\}_{t\in\mathbb{N}}$ is invertible at $\boldsymbol{\theta}$ and asymptotically stationary.

(b) The filtered volatility $\{\hat{\sigma}_t^2(\boldsymbol{\theta}, \hat{\sigma}_1^2)\}_{t\in\mathbb{N}}$ is not invertible at $\boldsymbol{\theta}$ and not asymptotically stationary.

(c) The filtered volatility $\{\hat{\sigma}_t^2(\boldsymbol{\theta}, \hat{\sigma}_1^2)\}_{t\in\mathbb{N}}$ is invertible at $\boldsymbol{\theta}$ but not asymptotically stationary.

(d) The filtered volatility $\{\hat{\sigma}_t^2(\boldsymbol{\theta}, \hat{\sigma}_1^2)\}_{t\in\mathbb{N}}$ is not invertible at $\boldsymbol{\theta}$ but it is asymptotically stationary.

Question 1 [28 points] - Multiple Choice (continued)

5. Consider the random sample $\{x_t\}_{t=1}^T$. Consider the extremum estimator

$$\hat{\theta}_T \in \arg\max_{\theta\in\Theta} \frac{1}{T}\sum_{t=1}^{T}(x_t - \theta) \quad \text{where } \Theta = [2, 10].$$

   (a) $\hat{\theta}_T$ is not consistent.
   (b) $\hat{\theta}_T \xrightarrow{p} 10$ as $T \to \infty$.
   (c) $\hat{\theta}_T = 2$ almost surely.
   (d) $\hat{\theta}_T = 10$ almost surely.

6. Consider the M-estimator $\hat{\theta}_T$ given by $\hat{\theta}_T \in \arg\max_{\theta\in\Theta} Q_T(\theta)$. Suppose that the estimator is consistent for $\theta_0 \in \Theta$, i.e. $\hat{\theta}_T \xrightarrow{p} \theta_0$ as $T \to \infty$.

   (a) $\hat{\theta}_T$ is asymptotically normal if $\sqrt{T}\hat{\theta}_T \xrightarrow{d} N(0, V)$, as $T \to \infty$, where V denotes the asymptotic variance of the estimator.
   (b) $\hat{\theta}_T$ is asymptotically normal if $\sqrt{T}\hat{\theta}_T \xrightarrow{d} N(\theta_0, V)$, as $T \to \infty$, where V denotes the asymptotic variance of the estimator.
   (c) $\hat{\theta}_T$ is asymptotically normal if $\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} N(\theta_0, V)$, as $T \to \infty$, where V denotes the asymptotic variance of the estimator.
   (d) $\hat{\theta}_T$ is asymptotically normal if $\sqrt{T}\hat{\theta}_T/V \xrightarrow{d} N(0, 1)$, as $T \to \infty$, where V denotes the asymptotic variance of the estimator.

7. Let $\mathbf{x}_T := (x_1, ..., x_T)$ be a subset of a strictly stationary and ergodic sequence $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^4 < \infty$. Consider the AR(1) model,

$$x_t = \theta x_{t-1} + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z}.$$

   You have decided to use the following estimator for the true parameter θ_0,

$$\hat{\theta}_T \in \arg\max_{\theta\in\Theta} -\frac{1}{T}\sum_{t=2}^{T}\left[\frac{u_t(\theta)^2}{1 + u_t(\theta)^2} + \left(\theta - 0.9\right)^2\right],$$

   where $u_t(\theta) = x_t - \theta x_{t-1}$

   (a) $\hat{\theta}_T$ cannot be consistent for θ_0.
   (b) $\hat{\theta}_T$ cannot be consistent if $\theta_0 = 0.9$.
   (c) If the least squares estimator $\tilde{\theta}_T \in \arg\max_{\theta\in\Theta} -\frac{1}{T}\sum_{t=2}^{T} u_t(\theta)^2$ is consistent, then $\hat{\theta}_T$ is also consistent.
   (d) $\hat{\theta}_T$ is always consistent for θ_0.

4 pts attributed for each correct answer in the multiple choice questions.

Answer:

1. - d
2. - c
3. - b
4. - a
5. - c
6. - No statement is correct.
7. - No statement is correct.

Question 2 [35 points]   Stochastic Properties of Nonlinear Dynamic Models

In economics and finance, time-series may sometimes exhibit time-varying conditional mean and volatility.

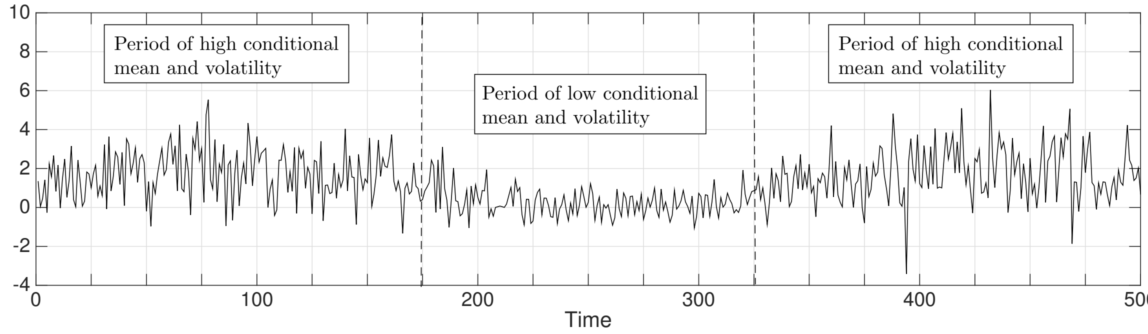

Figure 1: Time-series with time-varying conditional mean and volatility.

Let $\{x_t\}_{t\in\mathbb{Z}}$ be generated according to

$$x_t = \mu_t + \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of Gaussian iid random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1)$. Suppose that the time-varying conditional mean $\{\mu_t\}_{t\in\mathbb{Z}}$ satisfies

$$\mu_t = 0.2\big(x_{t-1} - \mu_{t-1}\big) + 0.7\mu_{t-1} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Furthermore, let the time-varying volatility $\{\sigma_t\}_{t\in\mathbb{Z}}$ be determined by an exogenous sequence $\{z_t\}_{t\in\mathbb{Z}}$, according to

$$\sigma_t = \big(1 + \tanh(z_t)\big) \quad \text{for every} \quad t \in \mathbb{Z}.$$

Finally, let $\{z_t\}_{t\in\mathbb{Z}}$ be generated by the following random coefficient autoregressive model

$$z_{t+1} = \rho_t z_t + v_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

where $\{\rho_t\}_{t\in\mathbb{Z}}$ is a sequence of iid random variables with uniform distribution $\{\rho_t\}_{t\in\mathbb{Z}} \sim \text{UID}(0,\ 1.5)$ taking values in the interval $[0\ ,\ 1.5]$, and $\{v_t\}_{t\in\mathbb{Z}}$ is a sequence of Student-t iid random variables with two degrees of freedom $\{v_t\}_{t\in\mathbb{Z}} \sim TID(2)$.

Note: the acronym *iid* stands for *independent identically distributed*.
Note: the function $1 + \tanh(\cdot)$ is uniformly bounded between 0 and 2.
Note: the random variable v_t satisfies $\mathbb{E}|v_t|^n < \infty$ for $0 < n < 2$.

(a) **(10pts)** Can you show that $\{\sigma_t\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic?

Answer: Since the tanh function is continuous, it is also measurable with respect to the Borel sigma-algebra. As a result, by Krengel's Theorem, we can conclude that $\{\sigma_t\}_{t\in\mathbb{Z}}$ is an SE sequence as long as $\{z_t\}_{t\in\mathbb{Z}}$ is itself SE.

Since $\{z_t\}_{t\in\mathbb{Z}}$ is generated by a Markov dynamical system,

$$z_{t+1} = \rho_t z_t + v_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

we can show that $\{z_t\}_{t\in\mathbb{Z}}$ is SE by application of Bourgerol's Theorem.

- Condition A1 of Bougerol's Theorem is satisfied since the innovation vector $\{(\rho_t, v_t)\}_{t\in\mathbb{Z}}$ is iid, and hence, it is trivially exogenous and SE.
- Condition A2 of Bougerol's Theorem is satisfied since, for any initialization z_1, we have

$$\begin{aligned}
\mathbb{E}\log^+|\rho_t z_1 + v_t| &\leq \mathbb{E}|\rho_t z_1 + v_t| \quad \text{(because } \log^+|x| \leq |x| \ \forall\ x\text{)}\\
&\leq |z_1|\mathbb{E}|\rho_t| + \mathbb{E}|v_t| \quad \text{(by sub-additivity of the absolute value)}\\
&\leq |z_1|1.5 + \mathbb{E}|v_t| \quad \text{(since } \{\rho_t\}_{t\in\mathbb{Z}} \sim \text{UID}(0\,,\ 1.5) \text{ and hence } |\rho_t| < 1.5\text{)}\\
&< \infty \quad \text{(because } z_1 \in \mathbb{R}\text{, and } \{v_t\}_{t\in\mathbb{Z}} \sim TID(2) \text{ and hence } \mathbb{E}|v_t|^n < \infty \text{ for any } n < 2\text{)}
\end{aligned}$$

- Condition A3 of Bougerol's Theorem (the contraction condition) is satisfied since $\partial(\rho_t z_t + v_t)/\partial z_t = \rho_t$ and hence

$$\begin{aligned}
\mathbb{E}\log\sup_z|\rho_t| &= \mathbb{E}\log|\rho_t|\\
&\leq \log\mathbb{E}|\rho_t| \quad \text{(Jensen's inequality)}\\
&= \log\mathbb{E}\rho_t \quad (\rho_t \text{ is always positive because } \{\rho_t\}_{t\in\mathbb{Z}} \sim \text{UID}(0\,,\ 1.5))\\
&= \log 0.75 < 0. \quad (\{\rho_t\}_{t\in\mathbb{Z}} \sim \text{UID}(0\,,\ 1.5) \text{ implies that } \mathbb{E}\rho_t = 0.75)
\end{aligned}$$

We thus conclude by Bougerol's Theorem that the process $\{z_t(z_1)\}_{t\in\mathbb{N}}$ initialized at any point z_1 converges to a limit SE sequence $\{z_t\}_{t\in\mathbb{Z}}$. Since Bougerol's conditions hold for any initialization $z_1 \in \mathbb{R}$, we can conclude that the sequence starting in the infinite past $\{z_t\}_{t\in\mathbb{Z}}$ is indeed SE.

Finally, as mentioned above, since the process starting in the infinite past $\{z_t\}_{t\in\mathbb{Z}}$ is SE, we conclude that $\{\sigma_t\}_{t\in\mathbb{Z}}$ is also SE.

(b) **(15pts)** Can you show that $\mathbb{E}|x_t|^2 < \infty$? Is $\{x_t\}_{t\in\mathbb{Z}}$ weakly stationary?

Answer: From the previous question (a), we already know that $\{\sigma_t\}_{t\in\mathbb{Z}}$ is SE. Furthermore, since the tanh function is uniformly bounded, we know that σ_t is also uniformly bounded

$$|\sigma_t| = |1 + \tanh(z_t)| \leq \sup_z|1 + \tanh(z)| \leq 2.$$

We will now show that $\{\mu_t\}_{t\in\mathbb{Z}}$ is SE and has two bounded moments by application of the Uniform Contraction Theorem with $n = 2$. First, we note that by substituting

x_{t-1} for $\mu_{t-1} + \sigma_{t-1}\varepsilon_{t-1}$ (observation equation) in the updating equation for μ_t, we find that $\{\mu_t\}_{t\in\mathbb{Z}}$ is generated according to

$$\mu_t = 0.2\sigma_{t-1}\varepsilon_{t-1} + 0.7\mu_{t-1} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Since this is a Markov dynamical system, we can simply verify if the conditions of the Uniform Contraction Theorem hold for $n = 2$

- Condition A1 of the Uniform Contraction Theorem holds because the innovations vector $\{(\sigma_{t-1}, \varepsilon_{t-1})\}_{t\in\mathbb{Z}}$ is clearly exogenous and SE. The SE nature of $\{\sigma_{t-1})\}_{t\in\mathbb{Z}}$ was shown in the previous question (a), and the SE nature of $\{\varepsilon_{t-1}\}_{t\in\mathbb{Z}}$ follows immediately since this sequence is iid.
- Condition A2 of the Uniform Contraction Theorem holds for any initialization μ_1 since

$$\begin{aligned}
\mathbb{E}|0.2\sigma_{t-1}\varepsilon_{t-1} + 0.7\mu_1|^2 &\leq 0.2c\mathbb{E}|\sigma_{t-1}\varepsilon_{t-1}|^2 + 0.7c\mathbb{E}|\mu_1|^2 \quad \text{(by the } c_n\text{-inequality)}\\
&\leq 0.2c\mathbb{E}|\sigma_{t-1}|^2|\mathbb{E}|\varepsilon_{t-1}|^2 + 0.7c|\mu_1|^2 \quad \text{(because } \sigma_{t-1} \text{ and } \varepsilon_{t-1} \text{ are independent)}\\
&\leq 0.2c2\mathbb{E}|\varepsilon_{t-1}|^2 + 0.7c|\mu_1|^2 \quad \text{(because } \sigma_t \text{ is uniformly bounded } \sigma_t < 2 \text{ a.s.)}\\
&< \infty \quad \text{(because } c \in \mathbb{R},\ \mu_1 \in \mathbb{R}, \text{ and } \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1) \text{ and hence } \mathbb{E}|\varepsilon_{t-1}|^n < \infty \forall n)
\end{aligned}$$

- Condition A3 of the Uniform Contraction Theorem (the contraction condition) is satisfied for $n = 2$ since

$$\sup_{\mu,\sigma,\varepsilon} \left| \frac{\partial(0.2\sigma_{t-1}\varepsilon_{t-1} + 0.7\mu)}{\partial\mu} \right|^2 = \sup_{\mu} |0.7|^2 = 0.7^2 < 1.$$

We thus conclude by application the Uniform Contraction Theorem, with $n = 2$, that the process $\{\mu_t(\mu_1)\}_{t\in\mathbb{N}}$ initialized at time $t = 1$ with some value μ_1 converges to a limit process $\{\mu_t\}_{t\in\mathbb{Z}}$ which is SE and has two bounded moments $\mathbb{E}|\mu_t|^2 < \infty$. Since, the conditions of the Uniform Contraction Theorem were shown to hold for any initialization $\mu_1 \in \mathbb{R}$ we further conclude that the process $\{\mu_t\}_{t\in\mathbb{Z}}$ initialized in the infinite past is indeed SE with two bounded moments.

Finally, we turn to the process $\{x_t\}_{t\in\mathbb{Z}}$. First, we note that $\{x_t\}_{t\in\mathbb{Z}}$ is SE by Krengel's Theorem since it is generated as a continuous function (and hence measurable w.r.t. the Borel sigma-algebra) of the independent SE sequences $\{\mu_t\}_{t\in\mathbb{Z}}$, $\{\sigma_t\}_{t\in\mathbb{Z}}$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$. Furthermore, we note that any element x_t of the sequence $\{x_t\}_{t\in\mathbb{Z}}$ has two bounded moments since

$$\begin{aligned}
\mathbb{E}|x_t|^2 &\leq \mathbb{E}|\mu_t + \sigma_t\varepsilon_t|^2\\
&\leq c\mathbb{E}|\mu_t|^2 + c\mathbb{E}|\sigma_t\varepsilon_t|^2 \quad \text{(by the } c_n\text{-inequality)}\\
&\leq c\mathbb{E}|\mu_t|^2 + c\mathbb{E}|\sigma_t|^2\mathbb{E}|\varepsilon_t|^2 \quad \text{(because } \sigma_t \text{ and } \varepsilon_t \text{ are independent)}\\
&\leq c\mathbb{E}|\mu_t|^2 + c4\mathbb{E}|\varepsilon_t|^2 \quad \text{(because } \sigma_t \leq 2 \text{ a.s.)}\\
&< \infty \quad \text{(because } c \in \mathbb{R},\ \mathbb{E}|\mu_t|^2 < \infty \text{ and } \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1) \text{ and hence } \mathbb{E}|\varepsilon_t|^n < \infty \forall n\text{)}
\end{aligned}$$

Since $\{x_t\}_{t\in\mathbb{Z}}$ is an SE sequence with two bounded moments $\mathbb{E}|x_t|^2 < \infty$, we can conclude that $\{x_t\}_{t\in\mathbb{Z}}$ is weakly stationary.

Question 3 [37 pts] - Estimation and Inference for Nonlinear Dynamic Models

Some econometricians claim that the temporal dependence in the growth rate of the *Gross Domestic Product* (GDP) is stronger during economic recession periods and weaker during expansions. Figure 1 plots the *growth rate of quarterly real GDP in The Netherlands.*

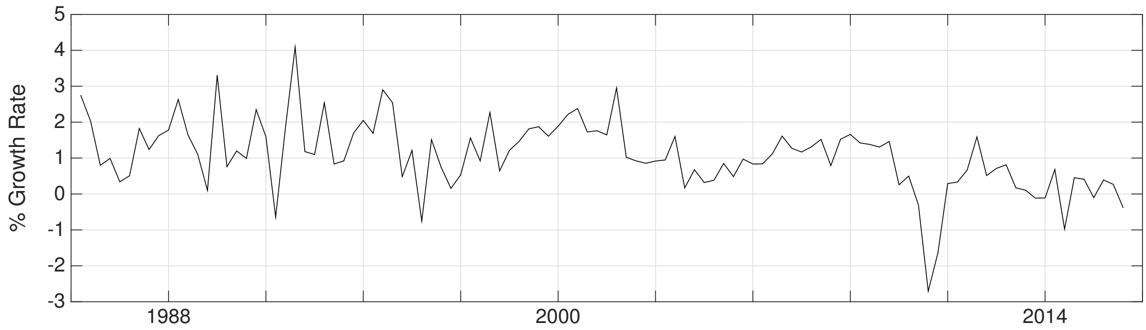

Figure 2: Real GDP growth rate for The Netherlands (in percentage).

Let the sample of GDP growth rates $\{x_t\}_{t=1}^T$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ with bounded moments of fourth order $\mathbb{E}|x_t|^4 < \infty$. Consider the Gaussian logistic *Self Excited Smooth Transition Autoregressive* (SESTAR) model

$$x_t = \alpha + g(x_{t-1};\boldsymbol{\theta})x_{t-1} + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0, \sigma_\varepsilon^2)$$

$$\text{and} \quad g(x_{t-1};\boldsymbol{\theta}) := \frac{\gamma}{1+\exp(\beta x_{t-1})} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Suppose that the parameters $\boldsymbol{\theta} = (\alpha, \gamma, \beta, \sigma_\varepsilon^2)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with $\sigma_\varepsilon^2 > 0$. Note also that $g(x;\boldsymbol{\theta})$ is uniformly bounded since $|g(x;\boldsymbol{\theta})| \leq |\gamma|$ for every $(x, \boldsymbol{\theta})$.

(a) (**15pts**) Suppose that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for a parameter $\boldsymbol{\theta}_0$ in the interior of Θ. Can you obtain an approximate distribution for the ML estimator?
Note: you can assume that certain functions are well behaved and continuously differentiable.

Answer: Note first that, according to the SESTAR model, the distribution of x_t conditional on x_{t-1} is given by

$$x_t|x_{t-1} \sim N\Big(\alpha + g(x_{t-1};\boldsymbol{\theta})x_{t-1}\ ,\ \sigma_\varepsilon^2\Big).$$

Hence, the conditional density of x_t given x_{t-1} takes the form,

$$f(x_t|x_{t-1};\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} \exp\left(\frac{-\big(x_t - \alpha - g(x_{t-1};\boldsymbol{\theta})x_{t-1}\big)^2}{2\sigma_\varepsilon^2}\right).$$

and the log likelihood function takes the form

$$Q_T(\boldsymbol{\theta}) := \frac{1}{T}\sum_{t=2}^{T} q(x_t, x_{t-1}, \boldsymbol{\theta}) \quad \text{where}$$

$$\begin{aligned} q(x_t, x_{t-1}, \boldsymbol{\theta}) &: = \log f(x_t|x_{t-1}; \boldsymbol{\theta}) \\ &= -\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma_\varepsilon^2 - \frac{(x_t - \alpha - g(x_{t-1}; \boldsymbol{\theta})x_{t-1})^2}{2\sigma_\varepsilon^2}. \end{aligned}$$

Since the ML estimator is consistent, i.e. $\hat{\boldsymbol{\theta}}_T \xrightarrow{p} \boldsymbol{\theta}_0$ as $T \to \infty$, to a point $\boldsymbol{\theta}_0$ in the interior of the parameter space Θ, we can apply the classical asymptotic normality theorem. In particular, if the following conditions hold,

C1. Asymptotic normality of the score

$$\sqrt{T}\frac{1}{T}\sum_{t=2}^{T} \nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0) \xrightarrow{d} N(0, \Sigma) \quad \text{as } T \to \infty.$$

C2. Uniform convergence of the second derivative

$$\sup_{\boldsymbol{\theta}\in\Theta} \|\frac{1}{T}\sum_{t=2}^{T} \nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}) - \mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta})\| \xrightarrow{p} 0 \quad \text{as } T \to \infty.$$

C3. Invertibility of $\mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}_0)$.

Then we can conclude that $\hat{\boldsymbol{\theta}}_T$ is asymptotically Gaussian

$$\sqrt{T}\Big(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\Big) \xrightarrow{d} N\Big(0, \Omega\Sigma\Omega^\top\Big) \quad \text{as } T \to \infty \text{ where } \Omega := \Big(\mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\Big)^{-1}.$$

The asymptotic normality of the score (Condition C.1) can be obtained by application of a Central Limit Theorem (CLT). In particular, we note that $\{\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\}_{t\in\mathbb{Z}}$ satisfies a CLT if:

C1.1. The sequence $\{\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\}_{t\in\mathbb{Z}}$ is SE.

C1.2. $\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)$ has two bounded moments $\mathbb{E}|\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)|^2 < \infty$.

C1.3. $\{\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\}_{t\in\mathbb{Z}}$ is either a martingale difference sequence, or it is a NED sequence.[1]

Condition C1.1 holds because q is continuously differentiable, and hence ∇q is continuous (and also measurable w.r.t. the Borel σ-algebra). This implies, by Krengel's Theorem, that $\{\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\}_{t\in\mathbb{Z}}$ is SE since ∇q is a measurable function of the SE sequence $\{x_t\}_{t\in\mathbb{Z}}$.

[1] The CLT for NED sequences requires r moments with $r > 2$, but as agreed in class, it is ok if you ignore this point!

Condition C1.2 holds since $q(x_t, x_{t-1}, \boldsymbol{\theta})$ has two bounded moments and the criterion function q is continuously differentiable and well behaved of order 2. The two bounded moments are obtained as follows

$$\begin{aligned}
\mathbb{E}|q(x_t, x_{t-1}, \boldsymbol{\theta})|^2 &= \mathbb{E}\Big| -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma_\varepsilon^2 - \frac{(x_t - \alpha - g(x_{t-1};\boldsymbol{\theta})x_{t-1}))^2}{2\sigma_\varepsilon^2}\Big|^2 \\
&\quad \text{(by the definition of } q(x_t, x_{t-1}, \boldsymbol{\theta})\text{)} \\
&\leq c\mathbb{E}\Big|\frac{1}{2}\log 2\pi\Big|^2 + c\mathbb{E}\Big|\frac{1}{2}\log\sigma_\varepsilon^2\Big|^2 + c\mathbb{E}\Big|\frac{(x_t - \alpha - g(x_{t-1};\boldsymbol{\theta})x_{t-1}))^2}{2\sigma_\varepsilon^2}\Big|^2 \\
&\quad \text{(by applying the } c_n\text{-inequality with some constant } c\text{)} \\
&\leq c(\frac{1}{2}\log 2\pi)^2 + c(\frac{1}{2}\log\sigma_\varepsilon^2)^2 + c\frac{1}{4\sigma_\varepsilon^4}\mathbb{E}\Big|x_t - \alpha - g(x_{t-1};\boldsymbol{\theta})x_{t-1})\Big|^4 \\
&\quad \text{(dropping the expectation for all constants)} \\
&\leq A + c\frac{1}{4\sigma_\varepsilon^4}\mathbb{E}\Big|x_t - \alpha - g(x_{t-1};\boldsymbol{\theta})x_{t-1})\Big|^4 \\
&\quad \text{(defining the constant } A := c(\tfrac{1}{2}\log 2\pi)^2 + c(\tfrac{1}{2}\log\sigma_\varepsilon^2)^2\text{)} \\
&\leq A + c'\frac{c}{4\sigma_\varepsilon^4}\mathbb{E}|x_t|^4 + c'\frac{c}{4\sigma_\varepsilon^4}|\alpha|^4 + c'\frac{c}{4\sigma_\varepsilon^4}\mathbb{E}|g(x_{t-1};\boldsymbol{\theta})x_{t-1}|^4 \\
&\quad \text{(applying again the } c_n\text{-inequality for some constant } c'\text{)} \\
&\leq A' + c'\frac{c}{4\sigma_\varepsilon^4}\mathbb{E}|x_t|^4 + c'\frac{c}{4\sigma_\varepsilon^4}\sup_x|g(x_{t-1};\boldsymbol{\theta})|^4\mathbb{E}|x_{t-1}|^4 \\
&\quad \text{(because } \sup_x|g(x;\boldsymbol{\theta})| \text{ is a constant it is taken outside the expectation)} \\
&\quad \text{(defining also } A' := A + c'\tfrac{c}{4\sigma_\varepsilon^4}|\alpha|^4\text{)} \\
&\leq A' + c'\frac{c}{4\sigma_\varepsilon^4}\mathbb{E}|x_t|^4 + c'\frac{c}{4\sigma_\varepsilon^4}\;\bar{k}\;\mathbb{E}|x_{t-1}|^4 \\
&\quad (\sup_x|g(x;\boldsymbol{\theta})| \leq \bar{k} \text{ for some } \bar{k}\in\mathbb{R} \text{ because } g(x;\boldsymbol{\theta}) \text{ is uniformly bounded)} \\
&< \infty. \\
&\quad \text{(because } A'\in\mathbb{R},\ c\in\mathbb{R},\ c'\in\mathbb{R},\ \sigma_\varepsilon^4 > 0 \Rightarrow 1/\sigma_\varepsilon^4\in\mathbb{R}, \text{ and } \mathbb{E}|x_t|^4 = \mathbb{E}|x_{t-1}|^4 < \infty\text{)}
\end{aligned}$$

Condition C1.3 will hold if appropriate conditions are satisfied:

(i) if the model is well specified, then $\{\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\}_{t\in\mathbb{Z}}$ is a martingale difference sequence;

(ii) if the model is misspecified, then $\{\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\}_{t\in\mathbb{Z}}$ will be NED if the data is NED (not stated in question) and if q is Lipschitz continuous.

We thus conclude that the score is asymptotically normal, i.e. that

$$\sqrt{T}\frac{1}{T}\sum_{t=2}^{T}\nabla q(x_t, x_{t-1}, \boldsymbol{\theta}_0) \xrightarrow{d} N(0,\Sigma) \quad \text{as } T\to\infty.$$

The uniform convergence of the second derivative (condition C2) can be obtained by ensuring that the following three conditions hold:

C2.1. Θ is compact.

C2.2. The second derivative satisfies a pointwise LLN

$$\frac{1}{T}\sum_{t=2}^{T} \nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}) \xrightarrow{p} \mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}) \;\forall\; \boldsymbol{\theta} \in \Theta \quad \text{as } T \to \infty.$$

C2.3. The second derivative is stochastically equicontinuous

$$\sup_t \mathbb{E} \sup_{\boldsymbol{\theta}\in\Theta} \left|\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})\right| < \infty.$$

The compactness of the parameter space (condition C2.1) holds by assumption.

The pointwise LLN (condition C2.2) holds since the following conditions are satisfied

C2.2.1 $\{\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta})\}_{t\in\mathbb{Z}}$ is SE for every $\boldsymbol{\theta} \in \Theta$.

C2.2.2 $\mathbb{E}|\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta})| < \infty$ for every $\boldsymbol{\theta} \in \Theta$.

The SE nature of the second derivative (condition C2.2.1) holds for every $\boldsymbol{\theta} \in \Theta$ because q is twice continuously differentiable, and hence $\nabla^2 q$ is continuous (and also measurable w.r.t. the Borel σ-algebra) for every $\boldsymbol{\theta} \in \Theta$. This implies, by Krengel's Theorem, that $\{\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta})\}_{t\in\mathbb{Z}}$ is SE for every $\boldsymbol{\theta} \in \Theta$ because $\nabla^2 q$ is a measurable function of the SE sequence $\{x_t\}_{t\in\mathbb{Z}}$ for every $\boldsymbol{\theta} \in \Theta$.

The moment bound for the second derivative (condition C2.2.2) holds since $q(x_t, x_{t-1}, \boldsymbol{\theta})$ has one bounded moment (see derivations above), the criterion function q is two times continuously differentiable, and both q and ∇q are well behaved of first order (WB(1)).

The stochastic equicontinuity of the criterion's second derivative (condition C2.3) holds because $q(x_t, x_{t-1}, \boldsymbol{\theta})$ has one bounded moment. First, since the criterion function q is three times continuously differentiable, we can conclude that $\{\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})\}_{t\in\mathbb{Z}}$ is SE by Krengel's Theorem, because $\nabla^3 q$ is continuous (and hence measurable w.r.t. Borel's σ-algebra) on the SE sequence $\{x_t\}_{t\in\mathbb{Z}}$. The SE nature of $\{\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})\}_{t\in\mathbb{Z}}$ implies also that

$$\sup_t \mathbb{E} \sup_{\boldsymbol{\theta}\in\Theta} \left|\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})\right| = \mathbb{E} \sup_{\boldsymbol{\theta}\in\Theta} \left|\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})\right|.$$

Second, since q, ∇q and $\nabla^2 q$ are well behaved of first order, the moment bound on the criterion $\mathbb{E}|q(x_t, x_{t-1}, \boldsymbol{\theta})| < \infty$ (shown above) ensures that $\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})$ has one bounded moment uniformly in Θ

$$\mathbb{E} \sup_{\boldsymbol{\theta}\in\Theta} \left|\nabla^3 q(x_t, x_{t-1}, \boldsymbol{\theta})\right| < \infty$$

and hence the second derivative is stochastically equicontinuous.

We have thus verified conditions C2.1, C2.2and C2.3 and are able to conclude that the second derivative converges uniformly (condition C2).

Finally, we turn to the invertibility condition C3. In particular, we obtain immediately that $\mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}_0)$ is invertible if $\boldsymbol{\theta}_0$ is the unique maximizer of the limit criterion function and the Hessian is regular. Uniqueness of $\boldsymbol{\theta}_0$ holds under appropriate conditions:

(i) if the model is correctly specified, then $\boldsymbol{\theta}_0$ is unique as long as it is identified by the *information inequality* theorem;

(ii) if the model is not correctly specified, then $\boldsymbol{\theta}_0$ may still be unique. If uniqueness fails, then invertibility of $\mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}_0)$ will fail and we cannot show asymptotic normality.

If conditions C1, C2 and C3, of the classical asymptotic normality theorem hold, we conclude that

$$\sqrt{T}\Big(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\Big) \xrightarrow{d} N\Big(0, \Omega\Sigma\Omega^\top\Big) \quad \text{as } T \to \infty \text{ where } \Omega := \Big(\mathbb{E}\nabla^2 q(x_t, x_{t-1}, \boldsymbol{\theta}_0)\Big)^{-1}.$$

This asymptotic result suggests that, in finite samples, we have

$$\sqrt{T}\Big(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\Big) \approx N\Big(0, \Omega\Sigma\Omega^\top\Big)$$

which naturally implies that $\hat{\boldsymbol{\theta}}_T$ is approximately Gaussian, with mean $\boldsymbol{\theta}_0$, and a vanishing variance-covariance matrix given by $\Omega\Sigma\Omega^\top/T$,

$$\hat{\boldsymbol{\theta}}_T \approx N\Big(\boldsymbol{\theta}_0\ ,\ \Omega\Sigma\Omega^\top/T\Big).$$

(b) (**10pts**) Explain how you can use the approximate distribution of $\hat{\boldsymbol{\theta}}_T$ to test the claim that the temporal dependence in the growth rate of Dutch GDP is stronger during economic recession periods and weaker during expansions.

Answer: Testing the hypothesis $\mathrm{H}_0 : \beta_0 = 0$ against the one-sided alternative that β_0 is strictly positive, $\mathrm{H}_1 : \beta > 0$, is probably the most natural way of addressing this question. Indeed, for $\beta = 0$, the SESTAR model reduces to the linear AR(1)

$$x_t = \alpha + \gamma x_{t-1} + \varepsilon_t.$$

This implies that the temporal dependence is the same during expansions and recessions under the null H_0. In contrast, under the alternative hypothesis H_1, the temporal dependence in the growth rate of Dutch GDP is stronger during economic recession periods and weaker during expansions (at least for a positive $\gamma > 0$ which the plotted data suggests). Hence, if we reject the null hypothesis that $\beta_0 = 0$ we favour the claim that *'the growth rate of Dutch GDP is stronger during economic recession periods and weaker during expansions'*.

In practice, we can use the approximate distribution derived in the previous questions as follows. If $H_0 : \beta = 0$ is true, then the asymptotic normality results tells us that the estimator $\hat{\beta}_T$ is approximately Gaussian with mean zero and some variance σ^2_β, i.e.

$$\hat{\beta}_T \approx N(0,\ \sigma^2_\beta).$$

After substituting σ_β^2 by some consistent estimator $\hat{\sigma}_\beta^2$, we can finally calculate the *approximate* tail probability of any given point estimate obtained from the data and judge how reasonable the hull hypothesis seems to be. The final decision depends on the adopted significance level.

Note: If the model is misspecified then the estimator $\hat{\sigma}_\beta^2$ should be robust to take into account the potential temporal dependence in the score. Furthermore, in this case, the null $\mathrm{H}_0 : \beta_0 = 0$ is a test on the MLE's pseudo-true parameter. The test then asks essentially if the best approximation to the DGP (in KL divergence, over the space of probability measures for the data) is delivered by a linear AR(1) model (H_0), or by a nonlinear SESTAR (H_1).