Preparation Exam
Advanced Econometrics (4.1)

Master Econometrics and Operations Research
Faculty of Economics and Business Administration

Exam:	Advanced Econometrics (4.1)
Code:	E_EORM_AECTR
Coordinator:	dr. F. Blasques
Date:	–
Time:	–
Duration:	2 hours and 45 minutes
Calculator:	Not allowed
Graphical calculator:	Not allowed
Number of questions:	4
Type of questions:	Open
Answer in:	English
Credit score:	100 credits counts for a 10
Grades:	Made public within 10 working days
Inspection:	By appointment (send e-mail to f.blasques@vu.nl)
Number of pages:	5, including front page

- Read the entire exam carefully before you start answering the questions.
- Be clear and concise in your statements, but justify every step in your derivations.
- If you think that further information is needed to answer a question or that the question is ill-posed, then explain your reasoning.
- The questions should be handed back at the end of the exam. Do not take it home.

Good luck!

Question 1 [25 points] Stochastic Properties of Nonlinear Dynamic Models

In economics and finance, time-series may sometimes exhibit time-varying conditional mean and volatility.

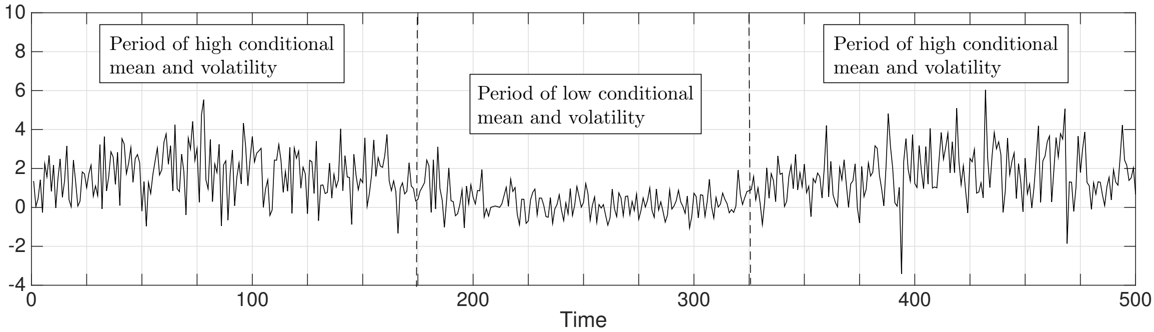

Figure 1: Time-series with time-varying conditional mean and volatility.

Let $\{x_t\}_{t\in\mathbb{Z}}$ be generated according to

$$x_t = \mu_t + \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of Gaussian iid random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1)$. Suppose that the time-varying conditional mean $\{\mu_t\}_{t\in\mathbb{Z}}$ satisfies

$$\mu_t = 0.2\big(x_{t-1} - \mu_{t-1}\big) + 0.7\mu_{t-1} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Furthermore, let the time-varying volatility $\{\sigma_t\}_{t\in\mathbb{Z}}$ be determined by an exogenous sequence $\{z_t\}_{t\in\mathbb{Z}}$, according to

$$\sigma_t = \big(1 + \tanh(z_t)\big) \quad \text{for every} \quad t \in \mathbb{Z}.$$

Finally, let $\{z_t\}_{t\in\mathbb{Z}}$ be generated by the following random coefficient autoregressive model

$$z_{t+1} = \rho_t z_t + v_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

where $\{\rho_t\}_{t\in\mathbb{Z}}$ is a sequence of iid random variables with uniform distribution $\{\rho_t\}_{t\in\mathbb{Z}} \sim \text{UID}(0,\ 1.5)$ taking values in the interval $[0\ ,\ 1.5]$, and $\{v_t\}_{t\in\mathbb{Z}}$ is a sequence of Student-t iid random variables with two degrees of freedom $\{v_t\}_{t\in\mathbb{Z}} \sim TID(2)$.

Note: the acronym *iid* stands for *independent identically distributed.*
Note: the function $1 + \tanh(\cdot)$ is uniformly bounded between 0 and 2.
Note: the random variable v_t satisfies $\mathbb{E}|v_t|^n < \infty$ for $0 < n < 2$.

(a) Can you show that $\{\sigma_t\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic?

(b) Can you show that $\mathbb{E}|x_t|^2 < \infty$? Is $\{x_t\}_{t\in\mathbb{Z}}$ weakly stationary?

Question 2 [25 points] Consistency and Asymptotic Normality of M-Estimators

Let $\mathbf{x}_T := (x_1, ..., x_T)$ be a subset of a fat-tailed strictly stationary and ergodic sequence $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^4 < \infty$. It is well known that the least squares estimator is sensitive to the presence of outliers in the data. Let $\hat{\boldsymbol{\theta}}_T$ be a robust M-estimator given by

$$\hat{\boldsymbol{\theta}}_T \in \arg\max_{\boldsymbol{\theta}\in\Theta} -\frac{1}{T}\sum_{t=2}^{T} \frac{u_t(\boldsymbol{\theta})^2}{1+u_t(\boldsymbol{\theta})^2}$$

where $u_t(\boldsymbol{\theta})$ denotes the regression residuals of a nonlinear autoregressive model

$$u_t(\boldsymbol{\theta}) := x_t - \phi(x_{t-1}, \boldsymbol{\theta}) \quad \text{for every } t.$$

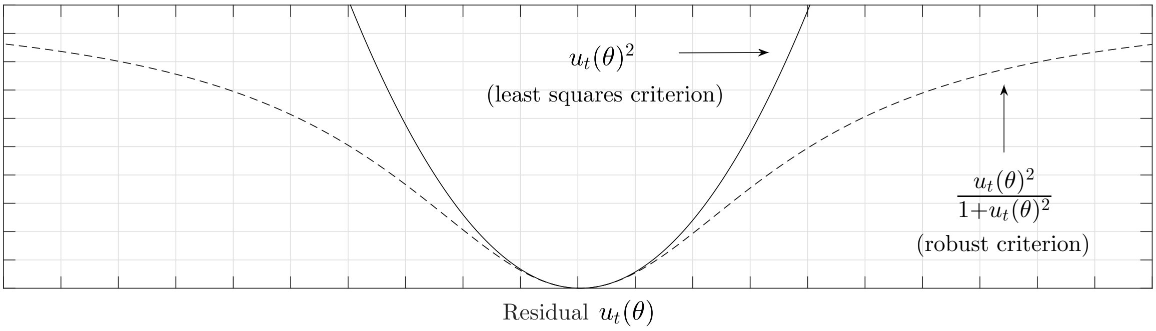

Figure 2: Comparison of the least squares and robust least squares criterion function.

Note: The function $u_t(\boldsymbol{\theta})^2/(1+u_t(\boldsymbol{\theta})^2)$ is uniformly bounded between 0 and 1.

(a) Give sufficient conditions for the existence and measurability of the estimator $\hat{\boldsymbol{\theta}}_T$.

(b) Give sufficient conditions for $\hat{\boldsymbol{\theta}}_T$ to be consistent for some point $\boldsymbol{\theta}_0 \in \Theta$. In other words, give conditions that ensure $\hat{\boldsymbol{\theta}}_T \xrightarrow{p} \boldsymbol{\theta}_0$ as $T \to \infty$.
Note: you can assume that certain functions are well behaved and continuously differentiable.

Question 3 [25 points] Nonlinear Dynamic Model of Dutch GDP

Some econometricians claim that the temporal dependence in the growth rate of the *Gross Domestic Product* (GDP) is stronger during economic recession periods and weaker during expansions. Figure 1 plots the *growth rate of quarterly real GDP in The Netherlands.*

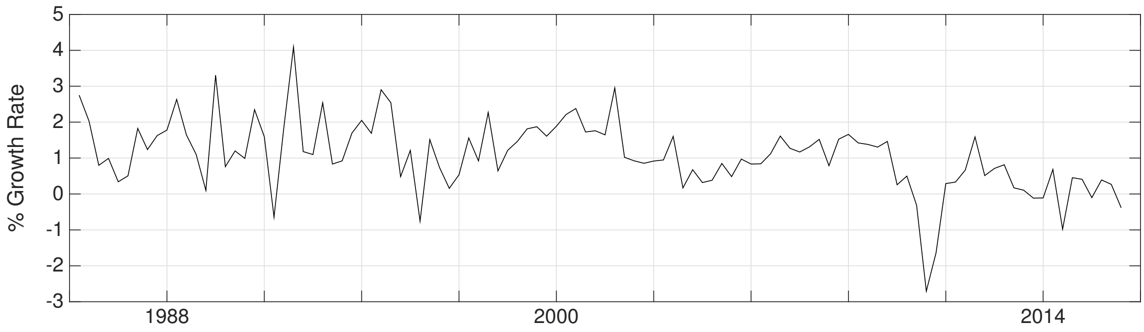

Figure 3: Real GDP growth rate for The Netherlands (in percentage).

Let the sample of GDP growth rates $\{x_t\}_{t=1}^T$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ with bounded moments of fourth order $\mathbb{E}|x_t|^4 < \infty$. Consider the Gaussian logistic *Self Excited Smooth Transition Autoregressive* (SESTAR) model

$$x_t = \alpha + g(x_{t-1};\boldsymbol{\theta})x_{t-1} + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0, \sigma_\varepsilon^2)$$

$$\text{and} \quad g(x_{t-1};\boldsymbol{\theta}) := \frac{\gamma}{1+\exp(\beta x_{t-1})} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Suppose that the parameters $\boldsymbol{\theta} = (\alpha, \gamma, \beta, \sigma_\varepsilon^2)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with $\sigma_\varepsilon^2 > 0$. Note also that $g(x;\boldsymbol{\theta})$ is uniformly bounded since $|g(x;\boldsymbol{\theta})| \leq |\gamma|$ for every $(x, \boldsymbol{\theta})$.

(a) Suppose that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for a parameter $\boldsymbol{\theta}_0$ in the interior of Θ. Can you obtain an approximate distribution for the ML estimator?
Note: you can assume that certain functions are well behaved and continuously differentiable.

(b) Explain how you can use the approximate distribution of $\hat{\boldsymbol{\theta}}_T$ to test the claim that the temporal dependence in the growth rate of Dutch GDP is stronger during economic recession periods and weaker during expansions.

Question 4 [25 points] Time-varying Conditional Volatility in Stock Markets

Financial returns often exhibit 'clusters of volatility' and 'leverage effects'. Figure 2 plots the time-series of *daily percentage returns* for the S&P500 stock market index.

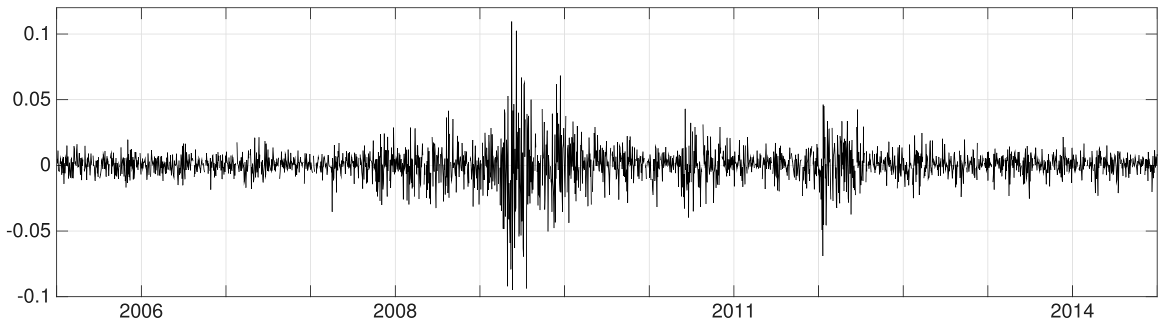

Figure 4: Daily S&P500 percentage returns.

Let the sample of S&P500 returns $\{x_t\}_{t=1}^{T}$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^8 < \infty$. Consider the *Asymetric Generalized Autoregressive Conditional Heteroscedasticity* (AGARCH) model

$$x_t = \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1),$$

$$\text{where} \quad \sigma_t^2 = \omega + \alpha(x_{t-1} - \delta)^2 + \beta\sigma_{t-1}^2 \quad \text{for every} \quad t \in \mathbb{Z}.$$

Suppose that the parameters $\boldsymbol{\theta} = (\omega, \alpha, \delta, \beta)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with ω, α and β satisfying

$$\omega > a, \quad \alpha > a, \quad \text{and} \quad a < \beta < 1 \quad \text{for some} \quad a > 0.$$

Note: that the parameter restrictions ensure that $\sigma_t^2 > a > 0$ for every t.

(a) Give the expression for the log likelihood function.

(b) Suppose that there exists a $\boldsymbol{\theta}_0 \in \Theta$ that is the unique maximizer of the limit log likelihood function. Can you show that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for $\boldsymbol{\theta}_0$?
Note: you can assume that certain functions are well behaved and continuously differentiable.