Exam Advanced Econometrics I (4.1)

Master Econometrics and Operations Research
Faculty of Economics and Business Administration
Monday, December 8, 2014

Exam:	Advanced Econometrics (4.1)
Code:	E_EORM_AECTR
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Co-Reader:	Prof. dr. Siem Jan Koopman
Date:	December 8, 2014
Time:	15:15
Duration:	2 hours and 45 minutes
Calculator:	Allowed
Graphical calculator:	Allowed
Number of questions:	4
Type of questions:	Open
Answer in:	English
Credit score:	100 credits counts for a 10
Grades:	Made public within 10 working days
Inspection:	By appointment (send e-mail to f.blasques@vu.nl)
Number of pages:	5, including front page

- Read the entire exam carefully before you start answering the questions.
- Be clear and concise in your statements, but justify every step in your derivations.
- If you think that further information is needed to answer a question or that the question is ill-posed, then explain your reasoning.
- The questions should be handed back at the end of the exam. Do not take it home.

Good luck!

Question 1 [25 points]   Stochastic Properties of Nonlinear Models

Let $\{x_t\}_{t\in\mathbb{Z}}$ be a random sequence generated according to

$$x_{t+1} = x_t \varepsilon_t^2 \quad \forall\, t \in \mathbb{Z} \ ,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of Gaussian independently distributed random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0, 0.5)$ with mean 0 and variance 0.5.

(a) Can you show that $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic?

(b) Can you show that $\mathbb{E}|x_t| < \infty$?

(c) Consider the statement: *"The time-series $\{x_t\}_{t\in\mathbb{Z}}$ is m-dependent"*. Is this statement true or false? Why is it true or false? Explain your reasoning.

Let $\{x_t\}_{t\in\mathbb{Z}}$ be a random sequence generated according to

$$x_t = 2.2 + 0.2\mu_t + \varepsilon_t \quad \forall\, t \in \mathbb{Z} \ ,$$

$$\text{with} \qquad \mu_t = 1.1 + 0.5(x_{t-1} - \mu_{t-1}) + 1.3\mu_{t-1} \quad \forall\, t \in \mathbb{Z} \ ,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of Student-t independently distributed random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{TID}(5)$ with 5 degrees of freedom.

(d) Can you show that $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic?

(e) Can you show that $\mathbb{E}|\mu_t|^2 < \infty$? And can you show that $\mathbb{E}|\mu_t|^6 < \infty$?
Note: $\mathbb{E}|\varepsilon_t|^n < \infty$ holds for $n < 5$.

(f) Consider the statement: *"The time-series $\{\mu_t\}_{t\in\mathbb{Z}}$ is weakly stationary but $\{x_t\}_{t\in\mathbb{Z}}$ is not"*. Is this statement true or false? Why is it true or false? Explain your reasoning.

Question 2 [20 points]   Consistency and Asymptotic Normality of M-Estimators

Let $\hat{\theta}_T$ be an M-estimator given by

$$\hat{\theta}_T \in \arg \max_{\theta \in [-1,1]} Q_T(\mathbf{x_T}, \theta)$$

$$\text{where} \quad Q_T(\mathbf{x_T}, \theta) := -\frac{1}{T} \sum_{t=2}^{T} \left(x_t - \frac{\theta}{1 + \exp(x_{t-1})} \right)^4 .$$

Suppose further that $\mathbf{x_T} := (x_1, ..., x_T)$ is a subset of a strictly stationary and ergodic sequence $\{x_t\}_{t \in \mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^4 < \infty$.

(a) Give sufficient conditions for the existence and measurability of the estimator $\hat{\theta}_T$.

(b) Can you show that the criterion function Q_T converges uniformly to some limit deterministic function Q_∞?

(c) Give sufficient conditions for $\hat{\theta}_T$ to be consistent for some point $\theta_0 \in \Theta$. In other words, give conditions that ensure $\hat{\theta}_T \xrightarrow{p} \theta_0$ as $T \to \infty$.

(d) Consider the following statement: *"The estimator $\hat{\theta}_T$ is strongly consistent for $\theta_0 \in \Theta$. In other words, the estimator satisfies $\hat{\theta}_T \xrightarrow{a.s.} \theta_0$ as $T \to \infty$".*

Question 3 [25 points] Nonlinear Dynamic Model of Real Exchange rates

Some econometricians claim that real exchange rates are stationary but exhibit strong temporal dependence and no mean-reverting behavior close to equilibrium. Figure 1 plots the real exchange rate of a EU 15 composite currency against the Danish Kroner.

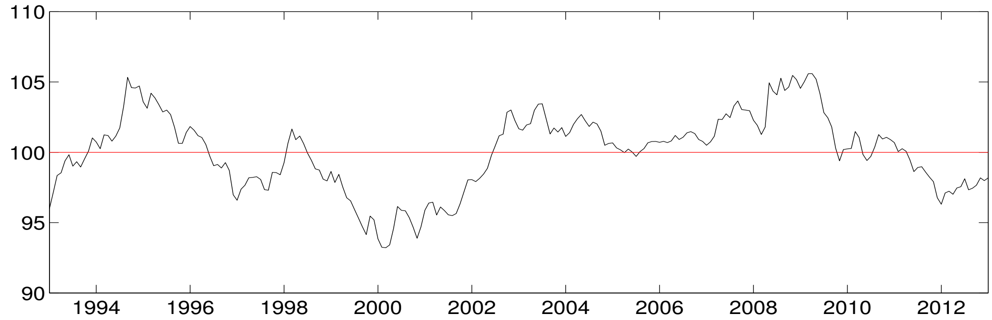

Figure 1: Real exchange rate of EU 15 vs Danish Kroner.

Let the sample of real exchange rates $\{x_t\}_{t=1}^T$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^4 < \infty$. Consider the *Gaussian Exponential SESTAR model*

$$x_t = \alpha + g(x_{t-1};\boldsymbol{\theta})x_{t-1} + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\epsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0, \sigma_\epsilon^2)$$

$$\text{and} \quad g(x_{t-1};\boldsymbol{\theta}) := \delta + \frac{\gamma}{1 + \exp(\beta(x_{t-1} - \mu)^2)} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Suppose that the parameters $\boldsymbol{\theta} = (\alpha, \delta, \gamma, \beta, \mu, \sigma_\varepsilon^2)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with $\sigma_\varepsilon^2 > 0$. Note also that $g(x;\boldsymbol{\theta})$ is uniformly bounded since $|g(x;\boldsymbol{\theta})| \leq |\delta| + |\gamma|$ for every $(x, \boldsymbol{\theta})$.

(a) Give the expression for the log likelihood function.

(b) Suppose that there exists a $\boldsymbol{\theta}_0 \in \Theta$ that is the unique maximizer of the limit log likelihood function. Can you show that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for $\boldsymbol{\theta}_0$?

Consider the AR(1) model

$$x_t = \omega + \rho x_{t-1} + v_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{v_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0, \sigma_v^2).$$

(c) Consider the statement: *"If the AR(1) model is well specified then the ML estimator for the parameters of the Gaussian exponential SESTAR model converges to the true parameter* $\boldsymbol{\theta}_0$*."*. Is this statement true or false? Why is it true or false? Explain your reasoning.

Question 4 [30 points] Time-varying Volatility in US Treasury Bills

Financial returns often exhibit 'clusters of volatility'. Figure 2 plots the time-series of weekly returns for the US treasury bill.

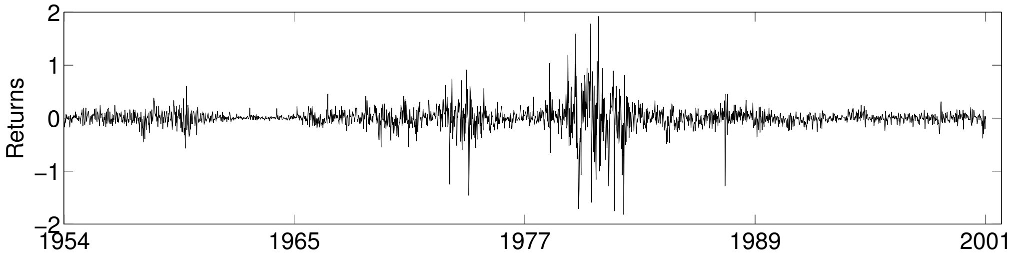

Figure 2: Weekly US treasury bills returns.

Let the sample of treasury bill returns $\{x_t\}_{t=1}^{T}$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^8 < \infty$. Consider the GARCH model for returns with mean δ,

$$x_t = \delta + \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1),$$

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta \sigma_{t-1}^2.$$

Suppose that the parameters $\boldsymbol{\theta} = (\omega, \alpha, \delta, \beta)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with ω, α and β satisfying

$$\omega > a, \quad \alpha > a, \quad \text{and} \quad a < \beta < 1 \quad \text{for some } a > 0.$$

Note that the parameter restrictions ensure that $\sigma_t^2 > a > 0$ for every t.

(a) Give the expression for the log likelihood function.

(b) Suppose that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for $\boldsymbol{\theta}_0$. Suppose that $\boldsymbol{\theta}_0$ is the unique maximizer of the limit log likelihood function in the interior of Θ. Can you obtain an approximate distribution for the ML estimator?
Note: you can assume that certain functions are well behaved and continuously differentiable.

(c) Explain how you can use the approximate distribution of $\hat{\boldsymbol{\theta}}_T$ to test the claim that financial returns exhibit 'clusters of volatility'.

(d) Consider the statement: *"If $\mathbb{E}\log|\alpha\epsilon_t^2 + \beta| < 0$ then the sequence $\{x_t\}_{t\in\mathbb{Z}}$ generated by the GARCH model is strictly stationary and ergodic".* Is this statement true or false? Why is it true or false? Explain your reasoning.