Exam Advanced Econometrics (4.1)

Master Econometrics and Operations Research
Faculty of Economics and Business Administration
Monday, October 20, 2014

Exam:	Advanced Econometrics (4.1)
Code:	E_EORM_AECTR
Coordinator:	dr. F. Blasques
Date:	October 20, 2014
Time:	12:00
Duration:	2 hours and 45 minutes
Calculator:	Allowed
Graphical calculator:	Allowed
Number of questions:	4
Type of questions:	Open
Answer in:	English
Credit score:	100 credits counts for a 10
Grades:	Made public within 10 working days
Inspection:	By appointment (send e-mail to f.blasques@vu.nl)
Number of pages:	5, including front page

- Read the entire exam carefully before you start answering the questions.
- Be clear and concise in your statements, but justify every step in your derivations.
- If you think that further information is needed to answer a question or that the question is ill-posed, then explain your reasoning.
- The questions should be handed back at the end of the exam. Do not take it home.

Good luck!

Question 1 [25 points] Stochastic Properties of Nonlinear Models

Let $\{x_t\}_{t\in\mathbb{Z}}$ be a random sequence generated according to

$$x_{t+1} = \beta_t x_t + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

where $\{\beta_t\}_{t\in\mathbb{Z}}$ is a sequence of uniformly independently distributed random variables $\{\beta_t\}_{t\in\mathbb{Z}} \sim \text{UID}(0,1)$ taking values in the interval $[0,1]$, and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of Gaussian independently distributed random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,\sigma_\varepsilon^2)$ with mean 0 and variance $0 < \sigma_\varepsilon^2 < \infty$.

(a) Can you show that $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic?

(b) Can you show that $\mathbb{E}|x_t|^2 < \infty$?

(c) Consider the statement: *"The time-series $\{x_t\}_{t\in\mathbb{Z}}$ is weakly stationary".* Is this statement true or false? Why is it true or false? Explain your reasoning.

Let $\{x_t\}_{t\in\mathbb{Z}}$ be a random sequence generated according to

$$x_t = \mu_t + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

$$\text{with} \qquad \mu_t = 2.5 + 0.2(x_{t-1} - \mu_{t-1}) + 0.7\mu_{t-1} \quad \text{for every} \quad t \in \mathbb{Z} \ ,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of Gaussian independently distributed random variables $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1)$ with mean 0 and variance 1.

(d) Can you show that $\{x_t\}_{t\in\mathbb{Z}}$ is strictly stationary and ergodic?

(e) Can you show that $\mathbb{E}|x_t|^8 < \infty$?

(f) Consider the statement: *"The time-series $\{x_t\}_{t\in\mathbb{Z}}$ is m-dependent".* Is this statement true or false? Why is it true or false? Explain your reasoning.

Question 2 [25 points] **Consistency and Asymptotic Normality of M-Estimators**

Let $\hat{\boldsymbol{\theta}}_T$ be an M-estimator given by

$$\hat{\boldsymbol{\theta}}_T \in \arg\max_{\boldsymbol{\theta}\in\Theta} Q_T(\mathbf{x_T}, \boldsymbol{\theta})$$

$$\text{where} \quad Q_T(\mathbf{x_T}, \boldsymbol{\theta}) := -\frac{1}{T}\sum_{t=2}^{T}\big(x_t - \phi(x_{t-1}, \boldsymbol{\theta})\big)^2.$$

Suppose further that $\mathbf{x_T} := (x_1, ..., x_T)$ is a subset of a strictly stationary and ergodic sequence $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^4 < \infty$.

(a) Give sufficient conditions for the existence and measurability of the estimator $\hat{\boldsymbol{\theta}}_T$.

(b) Suppose that Θ is compact and $\mathbb{E}|\phi(x_{t-1}, \boldsymbol{\theta})|^2 < \infty$. Can you show that the criterion function Q_T converges uniformly to some limit deterministic function Q_∞?
Note: you can assume that certain functions are well behaved and continuously differentiable.

(c) Give sufficient conditions for $\hat{\boldsymbol{\theta}}_T$ to be consistent for some point $\boldsymbol{\theta}_0 \in \Theta$. In other words, give conditions that ensure $\hat{\boldsymbol{\theta}}_T \xrightarrow{p} \boldsymbol{\theta}_0$ as $T \to \infty$.
Note: you can assume that certain functions are well behaved and continuously differentiable.

(d) Suppose further that $\mathbb{E}|\phi(x_{t-1}, \boldsymbol{\theta})|^4 < \infty$ and that $\hat{\boldsymbol{\theta}}_T$ is an estimator for a well specified model. Can you show that $\hat{\boldsymbol{\theta}}_T$ is asymptotically normal for the true parameter $\boldsymbol{\theta}_0$ in the interior of Θ? In other words, can you show that $\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{d} N(0, V)$ as $T \to \infty$?
Note: you can assume that certain functions are well behaved and continuously differentiable.

Question 3 [25 points] Nonlinear Dynamic Model of the Business Cycle

Some econometricians claim that the temporal dependence in the growth rate of GDP is stronger during economic recession periods and weaker during expansions. Figure 1 plots the *quarterly real GDP growth rate* in the *United States*.

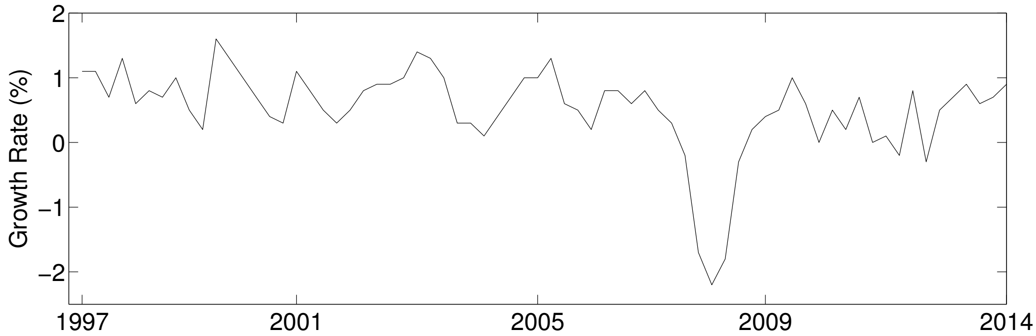

Figure 1: Real GDP growth rate for the US.

Let the sample of GDP growth rates $\{x_t\}_{t=1}^T$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ with bounded moments of fourth order $\mathbb{E}|x_t|^4 < \infty$. Consider the *Gaussian Logistic SESTAR model*

$$x_t = \alpha + g(x_{t-1};\boldsymbol{\theta})x_{t-1} + \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0, \sigma_\varepsilon^2)$$

$$\text{and} \quad g(x_{t-1};\boldsymbol{\theta}) := \delta + \frac{\gamma}{1+\exp(\beta x_{t-1})} \quad \text{for every} \quad t \in \mathbb{Z}.$$

Suppose that the parameters $\boldsymbol{\theta} = (\alpha, \delta, \gamma, \beta, \sigma_\varepsilon^2)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with $\sigma_\varepsilon^2 > 0$. Note also that $g(x;\boldsymbol{\theta})$ is uniformly bounded since $|g(x;\boldsymbol{\theta})| \leq |\delta| + |\gamma|$ for every $(x, \boldsymbol{\theta})$.

(a) Give the expression for the log likelihood function.

(b) Suppose that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for a parameter $\boldsymbol{\theta}_0$ in the interior of Θ. Can you obtain an approximate distribution for the ML estimator?
Note: you can assume that certain functions are well behaved and continuously differentiable.

(c) Explain how you can use the approximate distribution of $\hat{\boldsymbol{\theta}}_T$ to test the claim that the temporal dependence in the growth rate of GDP is stronger during economic recession periods and weaker during expansions.

Question 4 [25 points] Time-varying Volatility in US Treasury Bills

Financial returns often exhibit 'clusters of volatility' and 'leverage effects'. Figure 2 plots the time-series of *weekly returns* for the *US treasury bill.*

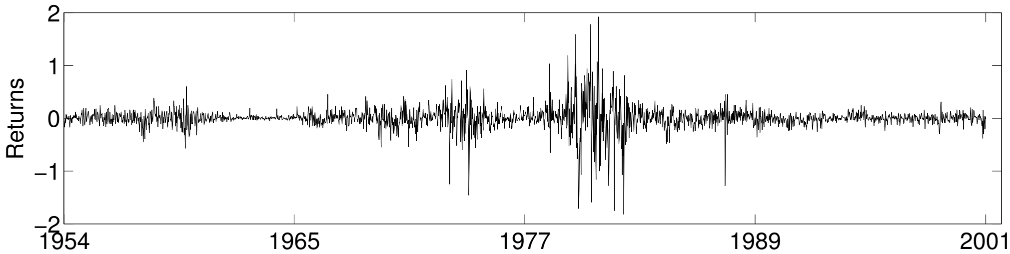

Figure 2: Weekly US treasury bills returns.

Let the sample of treasury bill returns $\{x_t\}_{t=1}^{T}$ at your disposal be a subset of the realized path of a strictly stationary and ergodic time-series $\{x_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}|x_t|^8 < \infty$. Consider the AGARCH model

$$x_t = \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1),$$

$$\text{where} \quad \sigma_t^2 = \omega + \alpha(x_{t-1} - \delta)^2 + \beta\sigma_{t-1}^2 \quad \text{for every} \quad t \in \mathbb{Z}.$$

Suppose that the parameters $\boldsymbol{\theta} = (\omega, \alpha, \delta, \beta)$ of the model are estimated by maximum likelihood (ML) on a compact parameter space Θ with ω, α, δ and β satisfying

$$\omega > a, \quad \alpha > a, \quad \text{and} \quad a < \beta < 1 \quad \text{for some} \quad a > 0.$$

Note that the parameter restrictions ensure that $\sigma_t^2 > a > 0$ for every t.

(a) Give the expression for the log likelihood function.

(b) Suppose that there exists a $\boldsymbol{\theta}_0 \in \Theta$ that is the unique maximizer of the limit log likelihood function. Can you show that the ML estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for $\boldsymbol{\theta}_0$?
Note: you can assume that certain functions are well behaved and continuously differentiable.

Consider the GARCH model

$$x_t = \sigma_t \varepsilon_t \quad \text{for every} \quad t \in \mathbb{Z} \quad \text{where} \quad \{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1)\ ,$$

$$\text{where} \quad \sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta\sigma_{t-1}^2 \quad \text{for every} \quad t \in \mathbb{Z}.$$

(c) Suppose that the ML estimators for both the AGARCH and the GARCH models are consistent. Consider the statement: *"If the AGARCH model is mis-specified then the ML estimator for the GARCH model converges to the pseudo-true parameter that ensures the best approximation to the conditional density of the random sequence of returns of the US treasury bill"*. Is this statement true or false? Why is it true or false? Explain your reasoning.