vrije Universiteit *amsterdam*
Faculty of Economics and Business Administration

Exam: Advanced Econometrics II

Code: E_EORM_AECTR

Coordinator: Michael Massmann

Date: 17 December 2012

Time: 15:15 hrs

Duration: 2 hours

Calculator allowed: No

Graphical calculator allowed: No

Number of questions: 2, with 4 parts each

Type of questions: Open

Answer in: English

Remarks:
*** all parts of all questions must be answered ***
*** the exam questions must not be taken out of the exam hall ***
*** this exam is scheduled for 2 hours, i.e. 120 minutes ***

Credit score: Total number of points: 100, maximum number of points per question: 50

Grades: The grades will be made public on: Monday 14 January 2013

Inspection: Wednesday 16 January 2013, at 16:00-17:00 hrs in room 1A-22

Number of pages: 4, including this front page

Good luck!

Question 1 (50 out of 100 points)

Consider the linear regression model with Normally distributed error terms:

$$y_t = X_t\beta + u_t, \quad \text{for } t = 1, \ldots, T \tag{1}$$

where y_t is a scalar random variable, X_t is a $(1 \times k)$-dimensional stochastic regressor, $u_t \mid X_t \sim \mathsf{NID}(0, \sigma_u^2)$, and $\beta, \sigma_u^2 \in \mathbb{R}^k \times \mathbb{R}^+$. The maximum likelihood estimator of $\theta' = (\beta', \sigma^2)$ is given by

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathsf{N}(0, \mathcal{I}_\infty(\theta_0)^{-1}) \tag{2}$$

where θ_0 is the true parameter point and $\mathcal{I}_\infty$ is the asymptotic information matrix.

(a) Derive the small-sample approximation of (2). Discuss the use of the information matrix (IM) and theoretical Hessian (TH) estimators of the small-sample variance-covariance matrix.

The equivalence between the IM and TH estimators in part **(a)** above is based on the asymptotic information matrix equality.

(b) Using the fact that the density of y_t, viz. $f_t(y_t; \theta) = f_t(\theta)$, integrates to unity:

$$\int f_t(\theta)\; dy_t = 1,$$

derive the asymptotic information matrix equality, discussing all steps carefully.

Suppose now that the error term u_t in (1) is autoregressive of order 1,

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad \text{for } t = 2, \ldots, T$$

where u_1 follows the stationary distribution of $\{u_t\}$, $\varepsilon_t \sim \mathsf{NID}(0, \sigma_\varepsilon^2)$, and $\rho, \sigma_\varepsilon^2 \in (-1, 1) \times \mathbb{R}^+$.

(c) Derive the exact sample loglikelihood function of this model. *NB:* You may use the fact that the density of a Normally distributed random variable z with mean μ and variance σ^2 is given by $f(z) = (2\pi\sigma^2)^{-1/2} \exp\{\frac{1}{2\sigma^2}(z - \mu)^2\}$.

Interest now centres on testing the hypothesis

$$H_0 : \rho = 0$$

against the alternative that

$$H_1 : \rho \neq 0.$$

(d) Explain carefully and in detail how you would proceed to implement *(i)* a Lagrange multiplier (LM) test, and *(ii)* an asymptotic F-test based on a Gauss-Newton regression to test H_0 against H_1.

Question 2 (50 out of 100 points)

Consider the following model in matrix notation:

$$y = Z_1\beta_1 + Y_1\beta_2 + u \tag{3}$$

where y and u are $(T \times 1)$ random vectors, Z_1 is a $(T \times k_1)$ matrix of exogenous regressors and Y_1 is a $(T \times k_2)$ matrix of endogenous explanatory variables. The reduced form of the latter is given by

$$Y_1 = Z\Pi + V_1 = Z_1\Pi_1 + Z_2\Pi_2 + V_1$$

where Z_2 is a $(T \times (l - k_1))$ matrix of exogenous regressors. Define also the $(T \times (k_1 + 1))$-dimensional matrix of endogenous variables $Y = [y : Y_1]$. Assume that $l - k_1 \geq k_2$.

(a) Define first Γ and B to write the system in the form

$$Y\Gamma = ZB + E$$

where $E = [u : V_1]$. Argue then why the sample loglikelihood function

$$\ell_T = -\frac{T}{2}\log(2\pi\Sigma) + T\,\text{logdet}(\Gamma) - \frac{1}{2}[\text{vec}(Y\Gamma - ZB)'(\Sigma \otimes I_T)^{-1}[\text{vec}(Y\Gamma - ZB] \tag{4}$$

with Σ being the contemporaneous variance-covariance matrix of E, may be simplified.

Concentrating the loglikelihood function in (4) sequentially with respect to all parameters but Γ yields

$$\tilde{\ell}_T(\Gamma) = c - \frac{T}{2}\log(\kappa) - \frac{T}{2}\text{logdet}(Y'M_ZY)$$

where

$$\kappa = \frac{(y - Y_1\beta_2)'M_{Z_1}(y - Y_1\beta_2)}{(y - Y_1\beta_2)'M_Z(y - Y_1\beta_2)}.$$

(b) Provide first a geometrical interpretation of the vectors $M_{Z_1}(y - Y_1\beta_2)$ and $M_Z(y - Y_1\beta_2)$ in the Euclidean space $\mathbb{E}^T$. Derive then the range of κ. Argue, finally, that when the equation in (3) is just-identified then $\kappa = 1$.

Assume that $k_1 = k_2 = 1$ and $l = 3$ such that $\Xi = [y : Y_1 : Z_1 : Z_2]$ is $(T \times 5)$-dimensional. Suppose that the (unscaled) sample second-moment matrix of the endogenous and exogenous variables is given by

$$\Xi'\Xi = \begin{pmatrix} 14 & 6 & 2 & 3 & 0 \\ 6 & 10 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

(c) Use the results in the appendix to show that the limited information maximum likelihood (LIML) estimate of κ is given by

$$\hat{\kappa} = 1.$$

Recall that $\kappa = 1$ when the equation in (3) is just-identified, as indeed was argued in part **(b)** above.

(d) Perform a likelihood ratio (LR) test for the validity of the overidentifying restrictions in the example in equation (5).

Useful results

Theorem 1 (least generalised variance ratio problem) *If A and B are symmetric and positive definite matrices of dimension p, and C is an arbitrary matrix of dimension $(p \times r)$ then*

$$\prod_{i=1}^{r} \lambda_i (A, B) \geq \frac{\det (C'AC)}{\det (C'BC)} \geq \prod_{i=r+1}^{p} \lambda_i (A, B) .$$

The upper bound is attained by choosing $C = V_{(r)}$ while the lower bound results if $C = V_{[r]}$, where V_i and $\lambda_i (A, B)$ are solutions to the generalised eigenvalue problem $AV = BV\Lambda$ while $V_{(r)}$ and $V_{[r]}$ denote the eigenvectors pertaining to the r largest and r smallest eigenvalues, respectively.

Note 1 *Denote by $\lambda(A, B)$ an eigenvalue of a generalised eigenvalue problem. The eigenvalue may be found by solving*

$$\det(A - \lambda B) = 0.$$