Please read carefully and start with answering those questions that you are sure you can solve. Write clearly, always state which exercise you are answering when using extra sheets, and indicate by crossing out if you want parts of your answer to be disregarded. Any ambiguity or unclarity will result in zero points, or in the worst of several solutions being graded. If you are uncertain of a formal statement or proof, write it down informally rather than giving no solution.

Note: you can write all solutions on the exam sheet, there is enough space reserved after each exercise. If you prefer or need extra space, you can also use exam paper. The exam has a total of 100 points.

Best Wishes!

Question 1 (12 points)

Which of the following statements are true given the assumptions in the following parts? List the implied statements and give a short reason for your decision.

a) (4 points) Let M be a stable matching between residents and hospitals, computed by the Gale-Shapley algorithm.

1) There exists no partial matching M' of any sets of residents R and hospitals H such that every vertex in R and H prefers their partner in M' over their partner in M.

2) As a participant in the matching process, it is more favorable to be on the side that accepts matching offers than on the side that has to propose.

b) (4 points) Let G be an undirected graph, where the A^* algorithm is used to compute a shortest path from vertex s to vertex t.

1) The algorithm considers no other edges of the graph than the ones that are part of a shortest path from s to t.

2) You have moved from house a to house s, and already know the best way to go from a to anywhere else in the city. $h(x) = dist(a, x) + dist(a, t)$ is a good heuristic to use in the algorithm, where $dist$ is the function representing your known routes from and to a.

c) (4 points) Assume that decision problem $L \in NP$.

1) Then, there exists no polynomial time reduction from SAT (the satisfyability problem) to L.

2) Then, it must be unknown whether there exists a polynomial time algorithm for L.

Question 2 (14 points)

Name two algorithmic paradigms/modeling techniques from the lecture that should be useful to solve the following problems optimally. For each, give an algorithm or a problem representation that does so (e.g., a greedy procedure, a formula for DynProg, a Flow Network) You do **not** need to state a proof of correctness.

Hint: *those techniques include Greedy, Dynamic Programming, Informed Search, Network Flows, Linear Programs, Integer Linear Programs.*

a) (6 points) You bought a new ventilation system for your house, which is very strong and can deliver fresh air to each of your n rooms R. However, you need to build a system of pipes such that for every room, there exists some pipe connection to the basement $b \in R$, where the ventilation system is located. Any pipe can be used for arbitrarily many rooms at the same time, and you know the costs for installing a pipe between any $(r_1, r_2) \in R \times R$. What is the minimum cost to get the ventilation system to work?

b) (8 points) The Knapsack Problem, i.e. given a number $K \geq 0$ and n items I with costs c_i and values v_i, choose a subset $S \subseteq I$ with $\sum_{i \in S} c_i \leq K$ that maximizes $\sum_{i \in S} v_i$.

Question 3 (8 points)
Recall that given an undirected graph $G = (V, E)$, a set $A \subseteq V$ is called *dominating* if for all $v \in V$ we have that $v \in A$ or there is a $u \in A$ such that $\{u, v\} \in E$. The DOMINATING SET problem is defined as follows: given a graph G and a natural number k, does G have a dominating set of size k?

The SET COVER problem is formulated as follows: Given a set U, a natural number n and a collection $\{S_1, S_2, \ldots, S_m\}$ with each $S_i \subseteq U$, is there a set of indices $I \subseteq \{1, 2, \ldots, m\}$ of size n such that $\bigcup_{i \in I} S_i = U$?

Prove that DOMINATING SET $\leq_p$ SET COVER .

Question 4 (14 points)
Recall that a propositional logic formula φ is in *2-CNF* if it is of the form $(\ell_{1,1} \vee \ell_{1,2}) \wedge (\ell_{2,1} \vee \ell_{2,2}) \wedge \cdots \wedge (\ell_{m,1} \vee \ell_{m,2})$, where each $\ell_{i,j}$ is a literal. Show that deciding whether there exists a satisfying assignment for a formula in *2-CNF* is in P, or prove that it is NP-hard.
Hint: *Finding a graph representation for the problem could help.*

Question 5 (10 points)

Design an algorithm that given a Graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbb{R}$, finds a matching of maximum weight: that is a set $A \subseteq E$ such that $w(A)$ is maximised. Analyse its runtime and argue for correctness, and in case of a non-optimal algorithm, prove an approximation bound.

Question 6 (8 points)

a) (2 points) Show $n^2 \notin \Omega(3n^3)$.

b) (2 points) Give the definition of a PTAS.

c) (2 points) Define the class NP.

d) (2 points) Is it known whether the maximum matching problem is NP-hard? Give a short reason.

Question 7 (10 points)

a) (5 points) Give an input instance (with n vertices, for any $n \in \mathbb{N}_{>2}$) to the k-center problem such that the greedy algorithm from the lecture, given that it starts with picking the first point in the input set V, always yields an optimal solution.

b) (5 points) Give a running time analysis for the k-center algorithm from the lecture (in general, not for the special instance constructed here).

Question 8 (13 points)

Given two numbers $2 \leq a \leq b$, and two operations: P (for Plus 1) and D (for Double), where $P(x) = x + 1$ and $D(x) = 2x$. Design a greedy algorithm to find the least amount of operations, starting from a, to get to b. Show its correctness. For example, if $a = 2$ and $b = 10$, the order would be DPD: $D(a) = 4, P(4) = 5$, and $D(5) = 10$.

Question 9 (14 points)
A company has r customers and q consultants. Every customer needs to be assigned to exactly one consultant. Every consultant can only be assigned to z customers. Finally, no consultant is allowed to drive more than 50 kilometers to any of their assigned customers; you may assume a table with all customer-consultant distances is given. You want to decide if there is a consultant-customer assignment that meets all of these demands.

a) (8 points) Model this problem as a maximum flow problem. Describe the graph you construct and argue why there exists a flow of value r in this graph if and only if a consultant-customer assignment exists.

b) (6 points) Argue how this problem could be solved by finding a bipartite matching. Describe the graph you construct.